

Abstract Harmonic Analysis on the General Linear Group $GL(n, \mathbb{R})$

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Abstract

Let $GL(n, \mathbb{R})$ be the general linear group and let $SL(n, \mathbb{R}) = KAN$ be the Iwasawa decomposition of real connected semisimple Lie group $SL(n, \mathbb{R})$. We adopt the technique in my paper [12] to generalize the definition of the Fourier transform and to obtain the Plancherel theorem for the group $SL(n, \mathbb{R})$. Since $GL_+(n, \mathbb{R}) = SL(n, \mathbb{R}) \times \mathbb{R}_+^*$ is the direct product of $SL(n, \mathbb{R})$ and \mathbb{R}_+^* , so we obtain the Plancherel formula for theorem $GL_+(n, \mathbb{R})$. In the end we prove that $GL_-(n, \mathbb{R})$ has a structure of group, which is isomorphic onto $GL_+(n, \mathbb{R})$, and leads us to obtain the Plancherel theorem for $GL(n, \mathbb{R})$.

Keywords: Linear Group $GL(n, \mathbb{R})$, Semisimple Lie Groups $SL(n, \mathbb{R})$, Fourier Transform, Plancherel Theorem

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1 Introduction.

1.1. As a manifold, $GL(n, \mathbb{R})$ is not connected but rather has two connected components, the matrices with positive determinant and the ones with nega-

tive determinant. The identity component, denoted by $GL_+(n, \mathbb{R})$, consists of the real $n \times n$ matrices with positive determinant. This is also a Lie group of dimension n^2 ; it has the same Lie algebra as $GL(n, \mathbb{R})$. The group $GL(n, \mathbb{R})$ is also noncompact. The maximal compact subgroup of $GL(n, \mathbb{R})$ is the orthogonal group $O(n)$, while the maximal compact subgroup of $GL_+(n, \mathbb{R})$ is the special orthogonal group $SO(n)$. As for $SO(n)$, the group $GL_+(n, \mathbb{R})$ is not simply connected (except when $n = 1$), but rather has a fundamental group isomorphic to \mathbb{Z} for $n = 2$ or \mathbb{Z}_2 for $n > 2$. Various physical systems, such as crystals and the hydrogen atom, can be modelled by symmetry groups. In quantum mechanics, we know that a change of frame a gauge transform leaves the probability of an outcome measurement invariant (well, the square modulus of the wave-function, i.e. the probability), because it is just a multiplication by a phase term theories of gravitation based, respectively, on the general linear group $GL(n, \mathbb{R})$ and its inhomogeneous extension $IGL(n, \mathbb{R})$. The topological groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ have an inexhaustibly rich structure and importance in all parts of modern mathematics: analysis, geometry, topology, representation theory, number theory, ect.... . The general linear group $GL(n, \mathbb{R})$ is decomposed into a Markov-type Lie group and an abelian scale group. The problem of finding the explicit Plancherel formulas for semisimple Lie groups has been solved completely in the case of complex semisimple Lie groups [13]. Moreover Harish-Chandra showed [15] that the problem is solved also for a real semisimple Lie group having only one conjugate class of Cartan subgroups. In the case of real semisimple Lie groups with several conjugate classes of Cartan subgroups, the problem is very difficult to attack. The problem was taken up and solved for $SL(2, \mathbb{R})$ by V. Bargman, [1], Harish-Chandra [14] and L. Pukanszky [21] also for the universal covering group of $SL(2, \mathbb{R})$ by L. Pukanszky. In this paper we will establish the Plancherel formula for the general linear group on $GL(n, \mathbb{R})$. The method is based on my paper [12]

2 Notation and Results

2.1. The fine structure of the nilpotent Lie groups will help us to do the Fourier transform on a simply connected nilpotent Lie groups N . As well

known any group connected and simply connected N has the following form

$$N = \begin{pmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & . & . & . & . & . & x_1^{n-2} & x_1^{n-1} & x_1^n \\ 0 & 1 & x_2^2 & x_2^3 & . & . & . & . & . & x_2^{n-2} & x_2^{n-1} & x_2^n \\ 0 & 0 & 1 & x_3^3 & . & . & . & . & . & x_3^{n-2} & x_3^{n-1} & x_3^n \\ 0 & 0 & 0 & 1 & . & . & . & . & . & x_4^{n-2} & x_4^{n-1} & x_4^n \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & x_{n-2}^{n-2} & x_{n-2}^{n-1} & x_{n-2}^n \\ 0 & 0 & 0 & 0 & . & . & . & . & . & 1 & x_{n-1}^{n-2} & x_{n-1}^{n-1} \\ 0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 1 & x_n^n \\ 0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

As shown, this matrix is formed by the subgroup $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^{n-1}$, and \mathbb{R}^n

$$\left(\mathbb{R} = \begin{bmatrix} x_1^1 \\ 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbb{R}^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ 1 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dots, \mathbb{R}^{n-1} = \begin{bmatrix} x_1^{n-1} \\ x_2^{n-1} \\ x_3^{n-1} \\ x_4^{n-1} \\ . \\ . \\ . \\ . \\ . \\ x_{n-2}^{n-1} \\ x_{n-2}^{n-2} \\ x_{n-1}^{n-1} \\ 1 \\ 0 \end{bmatrix}, \mathbb{R}^n = \begin{bmatrix} x_1^n \\ x_2^n \\ x_3^n \\ x_4^n \\ . \\ . \\ . \\ . \\ . \\ x_{n-2}^n \\ x_{n-1}^n \\ x_n^n \\ 1 \end{bmatrix} \right) \quad (2)$$

Each \mathbb{R}^i is a subgroup of N of dimension i , $1 \leq i \leq n$, put $d = n + (n - 1) + \dots + 2 + 1 = \frac{n(n+1)}{2}$, which is the dimension of N . According to [9], the group N is isomorphic onto the following group

$$((((\mathbb{R}^n \rtimes_{\rho_n} \mathbb{R}^{n-1}) \rtimes_{\rho_{n-1}} \mathbb{R}^{n-2} \rtimes_{\rho_{n-2}} \dots) \rtimes_{\rho_2} \mathbb{R}^2) \rtimes_{\rho_1} \mathbb{R} \quad (3)$$

That means

$$N \simeq (((((\mathbb{R}^n \rtimes_{\rho_n} \mathbb{R}^{n-1}) \rtimes_{\rho_{n-1}} \mathbb{R}^{n-2} \rtimes_{\rho_{n-2}} \dots) \rtimes_{\rho_4} \mathbb{R}^3 \rtimes_{\rho_3} \mathbb{R}^2) \rtimes_{\rho_2} \mathbb{R} \quad (4)$$

2.2. Denote by $L^1(N)$ the Banach algebra that consists of all complex valued functions on the group N , which are integrable with respect to the Haar measure of N and multiplication is defined by convolution on N as follows:

$$g * f(X) = \int_N f(Y^{-1}X)g(Y)dY \quad (5)$$

for any $f \in L^1(N)$ and $g \in L^1(N)$, where $X = (X^1, X^2, X^3, \dots, X^{n-2}, X^{n-1}, X^n)$, $Y = (Y^1, Y^2, Y^3, \dots, Y^{n-2}, Y^{n-1}, Y^n)$, $X^1 = x_1^1$, $X^2 = (x_1^2, x_2^2)$, $X^3 = (x_1^3, x_2^3, x_3^3)$, \dots , $X^{n-2} = (x_1^{n-2}, x_2^{n-2}, x_3^{n-2}, x_4^{n-2}, \dots, x_{n-2}^{n-2})$, $X^{n-1} = (x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, x_4^{n-1}, \dots, x_{n-2}^{n-1}, x_{n-1}^{n-1})$, $X^n = (x_1^n, x_2^n, x_3^n, x_4^n, \dots, x_{n-2}^n, x_{n-1}^n, x_n^n)$ and $dY = dY^1 dY^2 dY^3, \dots, dY^{n-2} dY^{n-1} dY^n$ is the Haar measure on N and $*$ denotes the convolution product on N . We denote by $L^2(N)$ its Hilbert space. Let \mathcal{U} be the complexified universal enveloping algebra of the real Lie algebra \underline{g} of N ; which is canonically isomorphic to the algebra of all distributions on N supported by $\{0\}$, where 0 is the identity element of N . For any $u \in \mathcal{U}$ one can define a differential operator P_u on N as follows:

$$P_u f(X) = u * f(X) = \int_N f(Y^{-1}X)u(Y)dY \quad (6)$$

The mapping $u \rightarrow P_u$ is an algebra isomorphism of \mathcal{U} onto the algebra of all invariant differential operators on N

3 Fourier Transform and Plancherel Theorem on N .

3.1. For each $2 \leq i \leq n$, let

$$K_{i-1} = \left\{ \left\{ \underbrace{\left\{ \underbrace{\left\{ \mathbb{R}^n \right\}_{\times_{\rho_n}} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}}_{\times_{\rho_{n-1}}} \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \times \dots}_{\times_{\rho_{i+1}}} \times \mathbb{R}^i \times \mathbb{R}^i \right\}_{\times_{\rho_i}} \times \mathbb{R}^{i-1} \times \mathbb{R}^{i-1} \right\} \right.$$

be the group with the following law

$$(X..., X^i, X^i, U^{i-1}, X^{i-1}).(Y..., Y^i, Y^i, V^{i-1}, Y^{i-1}) \quad (7)$$

$$= [(X..., X^i, X^i).(\rho_i(X^i)(Y..., Y^i, Y^i))], U^{i-1} + V^{i-1}, X^{i-1} + Y^{i-1}$$

For any $X \in L_{i+1}, Y \in L_{i+1}, X^i \in \mathbb{R}^i, U^{i-1} \in \mathbb{R}^{i-1}, X^{i-1} \in \mathbb{R}^{i-1}, Y^i \in \mathbb{R}^i, V^{i-1} \in \mathbb{R}^{i-1}, Y^{i-1} \in \mathbb{R}^{i-1}$. So we get for $i = 2$

$$K_1 = \left\{ \left\{ \left\{ \underbrace{\left\{ \underbrace{\left\{ \mathbb{R}^n \right\}_{\times_{\rho_n}} \times \mathbb{R}^{n-1,1} \times_{\rho_{n-1}} \mathbb{R}^{n-1}}_{\times_{\rho_{n-1}}} \times \mathbb{R}^{n-2,1} \times_{\rho_3} \mathbb{R}^{n-2} \times \dots}_{\times_{\rho_3}} \times \mathbb{R}^{2,1} \times \mathbb{R}^2 \right\}_{\times_{\rho_2}} \times \mathbb{R}^{1,1} \times \mathbb{R}^1 \right\} \right.$$

and for $i = n$, we get

$$K_{n-1} = \left\{ \underbrace{\mathbb{R}^n}_{\rtimes_{\rho_n}} \right\} \times \mathbb{R}^{n-1} \rtimes_{\rho_n} \mathbb{R}^{n-1}$$

3.2. Let $M = \mathbb{R}^n \times \mathbb{R}^{n-1,1} \times \mathbb{R}^{n-2,1} \times \dots \times \mathbb{R}^{3,1} \times \mathbb{R}^{2,1} \times \mathbb{R}^{1,1} = \mathbb{R}^d$ be the Lie group, which is the direct product of $\mathbb{R}^{n,1} = \mathbb{R}^n$, $\mathbb{R}^{n-1,1} = \mathbb{R}^{n-1}$, $\mathbb{R}^{n-2,1} = \mathbb{R}^{n-2}$, ..., $\mathbb{R}^{3,1} = \mathbb{R}^3$, $\mathbb{R}^{2,1} = \mathbb{R}^2$ and $\mathbb{R}^{1,1} = \mathbb{R}^1$. Denote by $L^1(M)$ the Banach algebra that consists of all complex valued functions on the group M , which are integrable with respect to the Lebesgue measure on M and multiplication is defined by convolution on M as:

$$g *_c f(X) = \int_M f(X - Y)g(Y)dY \quad (8)$$

for any $f \in L^1(M)$, $g \in L^1(M)$, where $*_c$ signifies the convolution product on the abelian group M . We denote again by \mathcal{U} be the complexified universal enveloping algebra of the real Lie algebra \underline{m} of M ; which is canonically isomorphic to the algebra of all distributions on M supported by $\{0\}$, where 0 is the identity element of M . For any $u \in \mathcal{U}$ one can define a differential operator Q_u on M as follows:

$$Q_u f(X) = u *_c f(X) = f *_c u(X) = \int_M f(X - Y)u(Y)dY \quad (9)$$

The mapping $u \rightarrow Q_u$ is an algebra isomorphism of \mathcal{U} onto the algebra of all invariant differential operators (the algebra of all linear differential operators with constant coefficients) on M

The group N can be identified with the subgroup

$$N = \left\{ \underbrace{\left\{ \underbrace{\left\{ \underbrace{\mathbb{R}^n}_{\rtimes_{\rho_n}} \right\} \times \{0\}}_{\rtimes_{\rho_{n-1}}} \rtimes_{\rho_{n-1}} \mathbb{R}^{n-1} \right\} \times \{0\} \rtimes_{\rho_3} \mathbb{R}^{n-2} \times \dots}_{\rtimes_{\rho_2}} \right\} \times \{0\} \rtimes_{\rho_2} \mathbb{R} \quad (10)$$

of K_1 and M can be identified with the subgroup

$$M = \left\{ \underbrace{\left\{ \underbrace{\left\{ \underbrace{\mathbb{R}^n}_{\rtimes_{\rho_n}} \right\} \times \mathbb{R}^{n-1,1} \rtimes_{\rho_n} \{0\}}_{\rtimes_{\rho_{n-1}}} \right\} \times \mathbb{R}^{n-2,1} \rtimes_{\rho_{n-1}} \{0\} \times \dots}_{\rtimes_{\rho_2}} \right\} \times \mathbb{R}^{1,1} \rtimes_{\rho_2} \{0\}$$

In this paper, we show how the Fourier transform on the vector group \mathbb{R}^d can be generalized on N and obtain the Plancherel theorem.

Definition 3.1. For $1 \leq i \leq n$, let \mathcal{F}^i be the Fourier transform on \mathbb{R}^i and $0 \leq j \leq n-1$, let $\prod_{0 \leq l \leq j} \mathbb{R}^{n-l} = (..(((\mathbb{R}^n \rtimes_{\rho_n} \mathbb{R}^{n-1}) \rtimes_{\rho_{n-1}}) \mathbb{R}^{n-2} \rtimes_{\rho_{n-2}}) .. \rtimes_{\rho_{n-j}} \mathbb{R}^{n-j})$, and let $\prod_{0 \leq l \leq j} \mathcal{F}^{n-l} = \mathcal{F}^n \mathcal{F}^{n-1} \mathcal{F}^{n-2} \dots \mathcal{F}^{n-j}$, we can define the Fourier

$$\begin{aligned}
& \text{transform on } \prod_{0 \leq l \leq n-1} \mathbb{R}^{n-l} = \mathbb{R}^n \rtimes_{\rho_n} \mathbb{R}^{n-1} \rtimes_{\rho_{n-1}} \mathbb{R}^{n-2} \rtimes_{\rho_{n-2}} \dots \rtimes_{\rho_3} \mathbb{R}^2 \rtimes_{\rho_2} \mathbb{R}^1 ds \\
& \mathcal{F}^n \mathcal{F}^{n-1} \mathcal{F}^{n-2} \dots \mathcal{F}^2 \mathcal{F}^1 f(\lambda^n, \lambda^{n-1}, \lambda^{n-2}, \dots, \lambda^2, \lambda^1) \\
& = \int_N f(X^n, X^{n-1}, \dots, X^2, X^1) e^{-i \langle (\lambda^n, \lambda^{n-1}), (X^n, X^{n-1}), \dots, (\lambda^2, \lambda^1), (X^2, X^1) \rangle} \\
& \quad dX^n dX^{n-1} \dots dX^2 dX^1 \tag{11}
\end{aligned}$$

for any $f \in L^1(N)$, where $X = (X^n, X^{n-1}, \dots, X^2, X^1)$, $\mathcal{F}^d = \mathcal{F}^n \mathcal{F}^{n-1} \mathcal{F}^{n-2} \dots \mathcal{F}^2 \mathcal{F}^1$ is the classical Fourier transform on N , $dX = dX^n dX^{n-1} \dots dX^2 dX^1$, $\lambda = (\lambda^n, \lambda^{n-1}, \lambda^{n-2}, \dots, \lambda^2, \lambda^1)$, and

$$\begin{aligned}
& \langle (\lambda^n, \lambda^{n-1}), (X^n, X^{n-1}), \dots, (\lambda^2, \lambda^1), (X^2, X^1) \rangle = \sum_{i=1}^n X_i^n \lambda_i^n + \sum_{j=1}^{n-1} X_j^{n-1} \lambda_j^{n-1} + \\
& \dots + \sum_{i=1}^2 X_i^2 \lambda_i^2 + X^1 \lambda^1
\end{aligned}$$

Plancherels theorem 3.2. For any function $f \in L^1(N)$, we have

$$\begin{aligned}
& \int_N |\mathcal{F}^d f(\xi^n, X^{n-1}, X^{n-2}, \dots, X^2, X^1)|^2 dX dX^{n-1} dX^{n-2} \dots dX^2 dX^1 \\
& \int_N |\mathcal{F}^d f(\xi^n, \lambda^{n-1}, \lambda^{n-2}, \dots, \lambda^2, \lambda^1)|^2 d\xi^n d\lambda^{n-1} d\lambda^{n-2} \dots d\lambda^2 d\lambda^1 \tag{12}
\end{aligned}$$

Proof: To prove this theorem, we refer to [12].

4 Fourier Transform and Plancherel Theorem

on AN .

4.1. Let $G = SL(n, \mathbb{R})$ be the real semi-simple Lie group and let $G = KAN$ be the Iwasawa decomposition of G , where $K = SO(n, \mathbb{R})$, and

$$N = \begin{pmatrix} 1 & * & . & . & * \\ 0 & 1 & * & . & * \\ . & . & . & . & * \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 1 \end{pmatrix}, \quad (13)$$

$$A = \begin{pmatrix} a_1 & 0 & 0 & . & 0 \\ 0 & a_2 & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & 0 & a_n \end{pmatrix} \quad (14)$$

where $a_1.a_2....a_n = 1$ and $a_i \in \mathbb{R}_+^*$. The product AN is a closed subgroup of G and is isomorphic (algebraically and topologically) to the semi-direct product of A and N with N normal in AN .

Then the group AN is nothing but the group $S = N \rtimes_{\rho} A$. So the product of two elements X and Y by

$$(x, a)(m, b) = (x.\rho(a)y, a.b) \quad (15)$$

for any $X = (x, a_1, a_2, .., a_n) \in S$ and $Y = (m, b_1, b_2, .., b_n) \in S$. Let $dnda = dmda_1da_2..da_{n-1}$ be the right haar measure on S and let $L^2(S)$ be the Hilbert space of the group S . Let $L^1(S)$ be the Banach algebra that consists of all complex valued functions on the group S , which are integrable with respect to the Haar measure of S and multiplication is defined by convolution on S as

$$g * f = \int_S f((m, b)^{-1}(n, a))g(m, b)dmdb \quad (16)$$

where $dmdb = dmdb_1db_2..db_{n-1}$ is the right Haar measure on $S = N \rtimes_{\rho} A$.

In the following we prove the Plancherel theorem. Therefore let $T = N \times A$ be the Lie group of direct product of the two Lie groups N and A ,

and let $H = N \times A \times A$ the Lie group, with law

$$(n, t, r)(m, s, q) = (n\rho(r)m, ts, rq) \quad (17)$$

for all $(n, t, r) \in H$ and $(m, s, q) \in H$. In this case the group S can be identified with the closed subgroup $N \times \{0\} \times_\rho A$ of H and T with the subgroup $N \times A \times \{0\}$ of H

Definition 4.1. For every function f defined on S , one can define a function \tilde{f} on L as follows:

$$\tilde{f}(n, a, b) = f(\rho(a)n, ab) \quad (18)$$

for all $(n, a, b) \in H$. So every function $\psi(n, a)$ on S extends uniquely as an invariant function $\tilde{\psi}(n, b, a)$ on L .

Remark 4.1. The function \tilde{f} is invariant in the following sense:

$$\tilde{f}(\rho(s)n, as^{-1}, bs) = \tilde{f}(n, a, b) \quad (19)$$

for any $(n, a, b) \in H$ and $s \in H$.

Lemma 4.1. For every function $f \in L^1(S)$ and for every $g \in L^1(S)$, we have

$$g * \tilde{f}(n, a, b) = g *_c \tilde{f}(n, a, b) \quad (20)$$

$$\int_{\mathbb{R}^{n-1}} \mathcal{F}_A^{n-1} \mathcal{F}^d(g * \tilde{f})(\lambda, \mu, \nu) d\nu = \mathcal{F}_A^{n-1} \mathcal{F}^d \tilde{f}(\lambda, \mu, 0) \mathcal{F}_A^{n-1} \mathcal{F}^d g(\lambda, \mu) \quad (21)$$

for every $(n, a, b) \in H$, where $*$ signifies the convolution product on S with respect the variables (n, b) and $*_c$ signifies the commutative convolution product on B with respect the variables (n, a) , where \mathcal{F}_A^{n-1} is the Fourier transform on A .

Plancherel Theorem 4.1. For any function $\Psi \in L^1(S)$, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} |\mathcal{F}^d \mathcal{F}_A^{n-1} \Psi(\lambda, \mu)|^2 d\lambda = \int_{AN} |\Psi(X, a)|^2 dX \quad (22)$$

Proof: To prove this theorem, we refer to [12].

5 Fourier Transform and Plancherel Theorem on $SL(n, \mathbb{R})$.

5.1. In the following we use the Iwasawa decomposition of $G = SL(n, \mathbb{R})$, to define the Fourier transform and to get Plancherel theorem on $G = SL(n, \mathbb{R})$. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group G , which are integrable with respect to the Haar measure of G and multiplication is defined by convolution on G ,

$$\phi * f(g) = \int_G f(h^{-1}g)\phi(g)dg \quad (23)$$

Let $G = SL(n, \mathbb{R}) = KNA$ be the Iwasawa decomposition of G . The Haar measure dg on G can be calculated from the Haar measures dn ; da and dk on N ; A and K ; respectively, by the formula

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dadndk \quad (24)$$

Keeping in mind that $a^{-2\rho}$ is the modulus of the automorphism $n \rightarrow ana^{-1}$ of N we get also the following representation of dg

$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dadndk = \int_N \int_A \int_K f(nak)a^{-2\rho}dndadk \quad (25)$$

where $\rho = \dim N = \frac{n(n-1)}{2} = 1 + 2 + 3 + \dots + n - 2 + n - 1$. Furthermore, using the relation $\int_G f(g)dg = \int_G f(g^{-1})dg$, we receive

$$\int_K \int_A \int_N f(nak)a^{-2\rho}dndadk = \int_K \int_A \int_N f(kan)a^{2\rho}dndadk \quad (26)$$

Let \underline{k} be the Lie algebra of K and (X_1, X_2, \dots, X_m) a basis of \underline{k} , such that the both operators

$$\Delta = \sum_{i=1}^m X_i^2 \quad (27)$$

$$D_q = \sum_{0 \leq l \leq q} \left(- \sum_{i=1}^m X_i^2 \right)^l \quad (28)$$

are left and right invariant (bi-invariant) on K , this basis exist see [2, p.564]. For $l \in \mathbb{N}$, let $D^l = (1 - \Delta)^l$, then the family of semi-norms $\{\sigma_l, l \in \mathbb{N}\}$ such that

$$\sigma_l(f) = \left(\int_K |D^l f(y)|^2 dy \right)^{\frac{1}{2}}, \quad f \in C^\infty(K) \quad (29)$$

define on $C^\infty(K)$ the same topology of the Frechet topology defined by the semi-norms $\|X^\alpha f\|_2$ defined as

$$\|X^\alpha f\|_2 = \left(\int_K |X^\alpha f(y)|^2 dy \right)^{\frac{1}{2}}, \quad f \in C^\infty(K) \quad (30)$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, for all the above formula see [2, P.176 – 177] and [2, p.565]

Let \widehat{K} be the set of all irreducible unitary representations of K . If $\gamma \in \widehat{K}$, we denote by E_γ the space of representation γ and d_γ its dimension

Definition 5.1. *The Fourier transform of a function $f \in C^\infty(K)$ is defined as*

$$Tf(\gamma) = \int_K f(x) \gamma(x^{-1}) dx \quad (31)$$

where T is the Fourier transform on K

Theorem (A. Cerezo) 5.1. *Let $f \in C^\infty(K)$, then we have the inversion of the Fourier transform*

$$f(x) = \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[Tf(\gamma) \gamma(x)] \quad (32)$$

$$f(x^{-1}) = \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[Tf(\gamma) \gamma(x^{-1})] = \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[Tf(\gamma) \gamma^*(x)] \quad (33)$$

$$f(I_K) = \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[Tf(\gamma)] \quad (34)$$

and the Plancherel formula

$$\|f(x)\|_2^2 = \int |f(x)|^2 dx = \sum_{\gamma \in \widehat{K}} d_\gamma \|Tf(\gamma)\|_{H.S}^2 \quad (35)$$

for any $f \in L^1(K)$, where I_K is the identity element of K , see [2, P.562 – 563], where $\|Tf(\gamma)\|_{H.S}^2$ is the norm of Hilbert-Schmidt of the operator $Tf(\gamma)$.

Definition 5.2. For any function $f \in \mathcal{D}(G)$, we can define a function $\Upsilon(f)$ on $G \times K$ by

$$\Upsilon(f)(g, k_1) = \Upsilon(f)(kna, k_1) = f(gk_1) = f(knak_1) \quad (36)$$

for $g = kna \in G$, and $k_1 \in K$. The restriction of $\Upsilon(f) * \psi(g, k_1)$ on $K(G)$ is $\Upsilon(f) * \psi(g, k_1) \downarrow_{K(G)} = f(nak_1) \in \mathcal{D}(G)$, and $\Upsilon(f)(g, k_1) \downarrow_K = f(g, I_K) = f(kna) \in \mathcal{D}(G)$

Remark 5.1. The function $\Upsilon(f)$ is invariant in the following sense

$$\Upsilon(f)(gh, h^{-1}k_1) = f(gk_1) = f(knak_1) \quad (37)$$

Definition 5.3. Let f and ψ be two functions belong to $\mathcal{D}(G)$, then we can define the convolution of $\Upsilon(f)$ and ψ on $G \times K$ as

$$\begin{aligned} \Upsilon(f) * \psi(g, k_1) &= \int_G \Upsilon(f)(gg_2^{-1}, k_1) \psi(g_2) dg_2 \\ &= \int_K \int_N \int_A \Upsilon(f)(knaa_2^{-1}n_2^{-1}k^{-1}k_1) \psi(k_2n_2a_2) dk_2 dn_2 da_2 \end{aligned}$$

So we get

$$\begin{aligned} \Upsilon(f) * \psi(g, k_1) \downarrow_{K(G)} &= \Upsilon(f) * \psi(I_K na, k_1) \\ &= \int_K \int_N \int_A f(naa_2^{-1}n_2^{-1}k^{-1}k_1) \psi(k_2n_2a_2) dk_2 dn_2 da_2 \end{aligned}$$

where T is the Fourier transform on K , and I_K is the identity element of K . Denote by \mathcal{F} is the Fourier transform on AN

Definition 5.4. If $f \in \mathcal{D}(G)$ and let $\Upsilon(f)$ be the associated function to f , we define the Fourier transform of $\Upsilon(f)(g, k_1)$ by

$$\begin{aligned} T\mathcal{F}\Upsilon(f)(I_K, \xi, \lambda, \gamma) &= T\mathcal{F}\Upsilon(f)(I_K, \xi, \lambda, \gamma) \\ &= \int_K \int_N \int_A \sum_{\delta \in \hat{K}} d_\delta \text{tr} \left[\int_K \Upsilon(f)(kna, k_1) \delta(k^{-1}) dk \right] a^{-i\lambda} e^{-i\langle \xi, n \rangle} \gamma(k_1^{-1}) da dn dk_1 \\ &= \int_N \int_A \int_K \Upsilon(f)(I_K na, k_1) a^{-i\lambda} e^{-i\langle \xi, n \rangle} \gamma(k_1^{-1}) da dn dk_1 \\ &= \int_N \int_A \int_K f(nak_1) a^{-i\lambda} e^{-i\langle \xi, n \rangle} \gamma(k_1^{-1}) da dn dk_1 \end{aligned} \quad (38)$$

Theorem 5.2. (*Plancherel's Formula for the Group G*). For any function $f \in L^1(G) \cap L^2(G)$, we get

$$\begin{aligned}
\int |f(g)|^2 dg &= \int_K \int_N \int_A |f(kna)|^2 da dn dk \\
&= \sum_{\gamma \in \widehat{K}} d_\gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \|T\mathcal{F}f(\lambda, \xi, \gamma)\|_{H.S}^2 d\lambda d\xi \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \|T\mathcal{F}f(\lambda, \xi, \gamma)\|_{H.S}^2 d\lambda d\xi \quad (39)
\end{aligned}$$

and so if $f \in L^1(G)$, then the Fourier inversion is

$$\begin{aligned}
f(ank_1) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[T\mathcal{F}f((\lambda, \xi, \gamma)\gamma(k_1))] a^{i\lambda} e^{i\langle \xi, n \rangle} d\lambda d\xi \\
f(I_A I_N I_K) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr}[T\mathcal{F}f(\lambda, \xi, \gamma)] d\lambda d\xi \quad (40)
\end{aligned}$$

where I_A, I_N and I_K are the identity elements of A, N and K respectively, where \mathcal{F} is the Fourier transform on AN and T is the Fourier transform on K , and I_K is the identity element of K

Proof: First let $\overset{\vee}{f}$ be the function defined by

$$\overset{\vee}{f}(kna) = \overline{f((kna)^{-1})} = \overline{f(a^{-1}n^{-1}k^{-1})} \quad (41)$$

Then we have

$$\begin{aligned}
& \int |f(g)|^2 dg \\
&= \Upsilon(f) * \overset{\vee}{f}(I_K I_N I_A, I_{K_1}) \\
&= \int_G \Upsilon(f)(I_K I_N I_A(g_2^{-1}), I_{K_1}) \overset{\vee}{f}(g_2) dg_2 \\
&= \int_A \int_N \int_K \Upsilon(f)(a_2^{-1} n_2^{-1} k_2^{-1}, I_K) \overset{\vee}{f}(k_2 n_2 a_2) da_2 dn_2 dk_2 \\
&= \int_A \int_N \int_K f(a_2^{-1} n_2^{-1} k_2^{-1}) \overline{f((k_2 n_2 a_2)^{-1})} da_2 dn_2 dk_2 \\
&= \int_A \int_N \int_K |f(a_2 n_2 k_2)|^2 da_2 dn_2 dk_2 \tag{42}
\end{aligned}$$

In other hand

$$\begin{aligned}
& \Upsilon(f) * \overset{\vee}{f}(I_K I_N I_A, I_{K_1}) \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \mathcal{F}(\Upsilon(f) * \overset{\vee}{f})(I_K, \lambda, \xi, I_{K_1}) d\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \sum_{\delta \in \widehat{K}} d_\delta \text{tr}[T\mathcal{F}(\Upsilon(f) * \overset{\vee}{f})(\delta, \lambda, \xi, \gamma)] d\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \mathcal{F}(\Upsilon(f) * \overset{\vee}{f})(I_K, \lambda, \xi, k_1) \gamma(k_1^{-1}) dk_1 \right] d\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_N \int_A \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \mathcal{F}(\Upsilon(f) * \overset{\vee}{f})(I_k n a, k_1) \gamma(k_1^{-1}) dk_1 \right] \\
& a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndad\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_N \int_A \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K \Upsilon(f)(I_k n a a_2^{-1} n_2^{-1} k_2^{-1}, k_1) \overset{\vee}{f}(k_2 n_2 a_2) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
& a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndada_2 dn_2 d\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_N \int_A \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K \Upsilon(f)(a a_2^{-1} n n_2^{-1} k_2^{-1}, k_1) \overset{\vee}{f}(k_2 n_2 a_2) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
& a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndada_2 dn_2 d\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_N \int_A \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K f(a a_2^{-1} n n_2^{-1} k_2^{-1} k_1) \overset{\vee}{f}(k_2 n_2 a_2) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
& a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndada_2 dn_2 d\lambda d\xi \\
= & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_N \int_A \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K f(a a_2^{-1} n n_2^{-1} k_1) \overset{\vee}{f}(k_2 n_2 a_2) \gamma(k_2^{-1}) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
& a^{-i\lambda} e^{-i\langle \xi, n \rangle} dndada_2 dn_2 d\lambda d\xi
\end{aligned}$$

We have used the fact that

$$\int_A \int_N \int_K f(kna) dadndk = \int_N \int_A \int_K f(kan) a^{2\rho} dndadk \quad (43)$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_A \int_N \int_K f(kna) e^{-i\langle \xi, n \rangle} da dn dk d\xi \\
&= \int_{\mathbb{R}^d} \int_A \int_N \int_K f(kan) e^{-i\langle \xi, ana^{-1} \rangle} a^{2\rho} da dn dk d\xi \\
&= \int_{\mathbb{R}^d} \int_A \int_N \int_K f(kan) e^{-i\langle a\xi a^{-1}, n \rangle} a^{2\rho} da dn dk d\xi \\
&= \int_{\mathbb{R}^d} \int_A \int_N \int_K f(kan) e^{-i\langle \xi, n \rangle} da dn dk d\xi \tag{44}
\end{aligned}$$

where $\langle \xi, n \rangle = \sum_{i=1}^d \xi_i n_i$, $d\xi = d\xi_1 d\xi_2$, and $ana^{-1} = a(\xi)a^{-1} = a^{-2\rho}\xi$, then we

get

$$\begin{aligned}
& \Upsilon(f) * \overset{\vee}{f}(I_K I_N I_A, I_{K_1}) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_A \int_N \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K f(ank_1) \overset{\vee}{f}(k_2 n_2 a_2) \gamma(k_2^{-1}) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} a_2^{-i\lambda} e^{-i\langle \xi, n_2 \rangle} da da_2 dn dn_2 d\lambda d\xi \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_A \int_N \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K f(ank_1) \overline{f((k_2 n_2 a_2)^{-1})} \gamma(k_2^{-1}) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} a_2^{-i\lambda} e^{-i\langle \xi, n_2 \rangle} dn da dn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_A \int_N \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K f(ank_1) \overline{f((a_2^{-1} n_2^{-1} k_2^{-1})} \gamma(k_2^{-1}) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} a_2^{-i\lambda} e^{-i\langle \xi, n_2 \rangle} dn da dn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \int_A \int_N \int_A \int_N \sum_{\gamma \in \widehat{K}} d_\gamma \text{tr} \left[\int_K \int_K f(ank_1) \overline{f((a_2 n_2 k_2)} \gamma^*(k_2^{-1}) \gamma(k_1^{-1}) dk_1 dk_2 \right] \\
&\quad a^{-i\lambda} e^{-i\langle \xi, n \rangle} \overline{a_2^{-i\lambda}} \overline{e^{-i\langle \xi, n_2 \rangle}} dn da dn_2 da_2 d\lambda d\xi \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma T\mathcal{F}f(\lambda, \xi, \gamma) \overline{T\mathcal{F}f(\lambda, \xi, \gamma^*)} d\lambda d\xi \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \|T\mathcal{F}(f)(\lambda, \xi, \gamma)\|_{H.S}^2 d\lambda d\xi
\end{aligned}$$

6 Fourier Transform and Plancherel Theorem

on $GL(n, \mathbb{R})$.

6.1. New Group. Let $GL(n, \mathbb{R})$ be the general linear group consisting of all matrices of the form

$$GL(n, \mathbb{R}) = \{X = (a_{ij}), a_{ij} \in \mathbb{R}, 1 \leq i \leq n, 1 \leq j \leq n, \text{ and } \det A \neq 0\} \quad (45)$$

As a manifold, $GL(n, \mathbb{R})$ is not connected but rather has two connected components: the matrices with positive determinant and the ones with negative determinant which is denoted by $GL_-(n, \mathbb{R})$. The identity component,

denoted by $GL_+(n, \mathbb{R})$, consists of the real $n \times n$ matrices with positive determinant. This is also a Lie group of dimension n^2 ; it has the same Lie algebra as $GL(n, \mathbb{R})$.

The group $GL(n, \mathbb{R})$ is also noncompact. The maximal compact subgroup of $GL(n, \mathbb{R})$ is the orthogonal group $O(n)$, while the maximal compact subgroup of $GL_+(n, \mathbb{R})$ is the special orthogonal group $SO(n)$. As for $SO(n)$, the group $GL_+(n, \mathbb{R})$ is not simply connected

Theorem 6.1. $GL_-(n, \mathbb{R})$ is group isomorphic onto $GL_+(n, \mathbb{R})$

Proof: $GL_-(n, \mathbb{R})$ is the subset of $GL(n, \mathbb{R})$, which is defined as

$$GL_-(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}), \det A < 0\} \quad (46)$$

We supply $GL_-(n, \mathbb{R})$ by the law noted \bullet which is defined by

$$A \bullet B = I^- A.B \quad (47)$$

for any $A \in GL_-(n, \mathbb{R})$ and $B \in GL_-(n, \mathbb{R})$, where $.$ signifies the usual multiplication of two matrix in $GL_-(n, \mathbb{R})$ and I^- is the the following matrix defined as

$$A = ([a_{ij}]) \quad (48)$$

and

$$I^- = \begin{pmatrix} -1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 1 \end{pmatrix} \quad (49)$$

Then we have $A \bullet B \in GL_-(n, \mathbb{R})$, for any $A \in GL_-(n, \mathbb{R})$ and $B \in GL_-(n, \mathbb{R})$.

The identity element is I^- because if

$$A \bullet B = I^- . A.B = B \quad (50)$$

then we have $A = I^-$ and so

$$I^- \bullet B = B \bullet I^- \quad (51)$$

The law is associative, because

$$\begin{aligned} A \bullet (B \bullet C) &= I^- . A.(B \bullet C) = I^- . A.(I^- . B.C) \\ &= A.B.C = (I^- . A.B).(I^- . C) = (A \bullet B)(I^- . C) \\ &= I^- . (A \bullet B).C = (A \bullet B) \bullet C \end{aligned}$$

and it is easy to show the inverse of any element $A \in GL_-(n, \mathbb{R})$ is

$$I^- . A \quad (52)$$

Now consider the mapping $\varphi : A \longrightarrow B$ defined by

$$\varphi(A) = I^- . A \quad (53)$$

for any $A \in GL_-(n, \mathbb{R})$. Then we get

$$\begin{aligned} \varphi(A \bullet B) &= I^-(A \bullet B) = I^- . I^- . A . B \\ &= I^- . A . I^- . B = \varphi(A) . \varphi(B) \end{aligned} \quad (54)$$

It is obvious that φ is one-to-one and onto, so φ is a group isomorphism from $GL_-(n, \mathbb{R})$ onto $GL_+(n, \mathbb{R})$. As well known the group $GL_+(n, \mathbb{R})$ is isomorphic onto the direct product of the two groups $SL(n, \mathbb{R})$ and \mathbb{R}_+^* , i.e. $GL_+(n, \mathbb{R}) = SL(n, \mathbb{R}) \times \mathbb{R}_+^*$ and $GL(n, \mathbb{R}) = GL_-(n, \mathbb{R}) \cup GL_+(n, \mathbb{R}) = (SL(n, \mathbb{R}) \times \mathbb{R}_+^*) \cup (SL(n, \mathbb{R}) \times \mathbb{R}_+^*)$. Our aim result is

Plancherel theorem 6.2. *Let \mathcal{F}_+^* be the Fourier transform on $GL_+(n, \mathbb{R})$ then we get*

$$\begin{aligned} \int_{GL_+(n, \mathbb{R})} |f(g, t)|^2 dg \frac{dt}{t} &= \int_{\mathbb{R}_+^*} \int_K \int_N \int_A |f(kna, t)|^2 da dn dk \frac{dt}{t} \\ &= \sum_{\gamma \in \widehat{K}} d_\gamma \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \|\mathcal{F}_+^* T \mathcal{F} f(\lambda, \xi, \gamma, \eta)\|_{H.S}^2 d\lambda d\xi d\eta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \|\mathcal{F}_+^* T \mathcal{F} f(\lambda, \xi, \gamma, \eta)\|_{H.S}^2 d\lambda d\xi d\eta \end{aligned} \quad (55)$$

for any $f \in L^1(GL_+(n, \mathbb{R})) \cap L^2(GL_+(n, \mathbb{R}))$. The proof of theorem results immediately from theorem 5.2

Corollary 6.1. *Let f be a function belongs $L^1(GL(n, \mathbb{R})) \cap L^2(GL(n, \mathbb{R}))$*

$$\begin{aligned}
& \int_{GL(n, \mathbb{R})} |f(g, t)|^2 dg \frac{dt}{t} \\
&= \int_{GL_-(n, \mathbb{R}) \cup GL_+(n, \mathbb{R})} |f(g, t)|^2 dg \frac{dt}{t} = 2 \int_{GL_+(n, \mathbb{R})} |f(g, t)|^2 dg \frac{dt}{t} \\
&= 2 \int_{\mathbb{R}_+^*} \int_K \int_N \int_A |f(kna, t)|^2 da dn dk \frac{dt}{t} \\
&= 2 \sum_{\gamma \in \widehat{K}} d_\gamma \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \|\mathcal{F}_+^* T \mathcal{F} f(\lambda, \xi, \gamma, \eta)\|_{H.S}^2 d\lambda d\xi d\eta \\
&= 2 \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{n-1}} \sum_{\gamma \in \widehat{K}} d_\gamma \|\mathcal{F}_+^* T \mathcal{F} f(\lambda, \xi, \gamma, \eta)\|_{H.S}^2 d\lambda d\xi d\eta \tag{56}
\end{aligned}$$

Remark 6.1. this corollary explains the Fourier transform and Plancherel formula on the non connected Lie group $GL(n, \mathbb{R})$.

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